In a nutshell: An adaptive Euler-Heun method

Note: This technique was developed as a teaching tool to introduce the Dormand-Prince method by this author. It is not described anywhere else.

Given the initial-value problem (IVP)

$$y^{(1)}(t) = f(t, y(t))$$
$$y(t_0) = y_0$$

we would like to approximate the solution $y(t)$ on the interval $[t_0, t_f]$ with a maximum error of ε_{abs} per unit time. This algorithm uses Taylor series and iteration. We start with an initial $h > 0$, we will have both minimum and maximum step sizes h_{min} and h_{max}, respectively.

1. Let $k \leftarrow 0$.
2. If $t_k \geq t_f$, we are finished: we have approximated values for $y(t_1)$ through $y(t_k)$, and using cubic splines, we can approximate values at any point on the interval $[t_0, t_f]$.
3. If $k > N$, we will return signalling that too many steps were required to find the approximations.
4. Let $s_0 = f(t_k, y_k)$
   $s_1 = f(t_k + h, y_k + hs_0)$,

   and thus, let $y \leftarrow y_k + hs_0$ and $z \leftarrow y_k + h\frac{s_0 + s_1}{2}$. y and z both approximate $y(t_k + h)$ but z is more accurate[1]
5. Let $a \leftarrow \frac{h\varepsilon_{\text{abs}}}{2|y - z|}$. *ah* estimates the ideal step size
6. If $a > 1$ or $h = h_{\text{min}}$, we will set $t_{k+1} \leftarrow t_k + h$ and set $y_{k+1} \leftarrow z$ and then increment k.

   If the ideal step size is greater than our current step size, or if the step size is already the minimum we will allow it, use z to approximate $y(t_k + h)$
7. If $0.9a < ½$, update $h \leftarrow ½h$,
   if $0.9a > 2$, update $h \leftarrow 2h$,
   otherwise update $h \leftarrow 0.9ah$. Update h with $0.9ah$ unless this more than doubles or halves its value
8. If $h < h_{\text{min}}$, set $h \leftarrow h_{\text{min}}$, and
   if $h > h_{\text{max}}$, set $h \leftarrow h_{\text{max}}$. Don't let h exceed the lower or upper bounds we've set on it
9. Return to Step 2.

[1] Normally, nutshells don't have such comments, but they are included here for clarity.